PCMI topological aspects of quantum codes, problem session #3

Instructor: Jeongwan Haah, Teaching Assistant: John Bostanci

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1. **(Density Matrices and Linear Functionals.)** Let ρ_{AB} be a density matrix on a bipartite system and consider the linear functional on the space of Hermitian operators acting on A defined by
$$O_{\mathsf{A}} \mapsto \mathrm{Tr}((O_{\mathsf{A}} \otimes \mathrm{id}_{\mathsf{B}})\rho_{\mathsf{AB}}).$$
   (a) Does there exists a matrix σ_{A} such that $\mathrm{Tr}(O_{\mathsf{A}}\sigma_{\mathsf{A}}) = \mathrm{Tr}((O_{\mathsf{A}} \otimes \mathrm{id}_{\mathsf{B}})\rho_{\mathsf{AB}})$ for all O_{A}?
   (b) Is σ unique?
   (c) How do you compute σ?

   Solution: σ_{A} is result of tracing out the B register of ρ_{AB}.

2. **(Baker–Campbell–Hausdorff.)** Let H and A be square matrices of the same finite dimension. Show that the following equality holds:
$$e^{iHt}Ae^{-iHt} = \sum_{k=0}^{\infty}(it)^k\frac{[H,A]_k}{k!}.$$
   Here $[H,A]_k$ is the nested commutator, defined as
$$\begin{aligned}[H,A]_0 &= A\\ [H,A]_k &= [H,[H,A]_{k-1}].\end{aligned}$$
   Show that the right hand side converges absolutely for any $t \in \mathbb{C}$ (i.e. taking the sum of the norms of the terms in the original sum converges) and hence both sides are complex analytic in t.

3. **(Cats on a Line.)** Consider a cat state, $\frac{1}{\sqrt{2}}(|0,\ldots,0\rangle + |1,\ldots,1\rangle)$, where the qubits are arranged on a line.
   (a) Find two observables (one on each end of the line) that have non-zero correlation with each other.
   (b) Prove that the cat state on a line can not be prepared in constant depth.
   (c) Find a linear depth circuit using 1- and 2-qubit local gates that synthesizes the cat state.

   Solution:

   (a) We can take the observables to be Z on the first and last qubit. The product will always yield a $+1$ eigenvalue (i.e. either both 0 or both 1), but each individual one is $+1$ with a half probability.

(b) By a light cone argument, when we evolve by a constant depth geometrically local circuit, two observables that are separated by more than that constant apart must be un-correlated. Thus the cat state can not be prepared in constant depth.

(c) For a linear depth circuit, start by applying H to the first qubit, and then apply $CNOT$ between the i and $i+1$'th qubits, from $i = 0$ to $i = n-1$.

4. **(Twist Product.)** Recall the definition of the twisted product. Given operators O_{AB} and O'_{AB}, we can express O and O' as a sum of tensor product operators as follows

$$O = \sum_{i=0}^{k} (\Gamma_i)_{\mathsf{A}} \otimes (\Lambda_i)_{\mathsf{B}}$$

$$O' = \sum_{j=0}^{k'} (\Gamma'_j)_{\mathsf{A}} \otimes (\Lambda'_j)_{\mathsf{B}}$$

Note that the operators in the decomposition are not necessarily positive semi-definite. Then define the twist product as

$$O \infty O' = \sum_{i=0}^{k} \sum_{j=0}^{k'} (\Gamma_i \Gamma'_j)_{\mathsf{A}} \otimes (\Lambda'_j \Lambda_i)_{\mathsf{B}}. \tag{1}$$

(a) Show that the twist product is a bilinear map (i.e. for a fixed O', the map $O \mapsto O \infty O'$ is linear, and vice-versa).

(b) Show that the twist product is well defined, independent of the choice of decomposition of O and O' into a sum of tensor product operators.