PCMI topological aspects of quantum codes, problem session #1

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1. **(CSS code cleaning lemma.)** Prove a CSS code cleaning lemma: Let $\mathcal{S}$ be a CSS code over n qubits and $M \subset \Lambda$ be a subset of qubits such that every operator supported only on M is not a non-trivial logical X operator. Then there exists a choice of representatives of all logical Z operators such that the representatives are supported on $\Lambda \setminus M$.

2. **(Finishing up the quantum Singleton bound.)** In the proof of the quantum Singleton bound, show that for two parties that share a bipartite state ρ_{AB}, if for all pairs of Hermitian operators $O_{\mathsf{A}} \otimes \mathrm{id}_{\mathsf{B}}, \mathrm{id}_{\mathsf{A}} \otimes O_{\mathsf{B}}$,

$$\mathrm{Tr}((O_{\mathsf{A}} \otimes O_{\mathsf{B}})\rho_{\mathsf{AB}}) = \mathrm{Tr}((O_{\mathsf{A}} \otimes \mathrm{id}_{\mathsf{B}})\rho_{\mathsf{AB}}) \cdot \mathrm{Tr}((\mathrm{id}_{\mathsf{A}} \otimes O_{\mathsf{B}})\rho_{\mathsf{AB}}), \tag{1}$$

   then their mutual information is 0.

3. **(Subadditivity and Nonnegativity.)** Recall the definition of the von Neumann entropy, $S(\rho) = -\mathrm{Tr}(\rho \ln \rho) = -\sum_j \lambda_j \ln(\lambda_j)$, where λ_j is the j'th eigenvalue of ρ, let ρ_{AB} be a density matrix over registers A and B. Show that the Von Neumann entropy satisfies

$$S(\rho_{\mathsf{AB}}) \leq S(\rho_{\mathsf{A}}) + S(\rho_{\mathsf{B}}).$$

4. **(Codes on non-orientable surfaces.)** We have bounded the number of logical qubits on any code defined on a two-dimensional torus by dividing the torus into three regions, two of which are correctable. Each correctable region is a union of two disk-like regions where the r-neighborhood of any one of the disk-like regions is also disk-like. Under the same assumption on correctable regions, bound the number of logical qubits of codes on $\mathbb{R}P^2$. Can you generalize it to higher demigenus nonorientable surfaces?

3 Hint: You may assume that the quantum relative entropy, defined below, is always non-negative:

$$S(\rho \| \sigma) = \mathrm{Tr}(\rho(\ln \sigma - \ln \rho)).$$

4 Hint: A region does not have to be the union of two subregions. There can be more.